

The Cauchy problem for higher-order modified Camassa-Holm equations on the circle

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Abstract. In this paper, we investigate the Cauchy problem for the shallow water type equation

$$u_t + \partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) + \partial_x(1 - \partial_x^2)^{-1} \left[u^2 + \frac{1}{2}u_x^2 \right] = 0$$

with low regularity data in the periodic settings. Himonas and Misiolek (Communications in Partial Differential Equations, 23(1998), 123-139.) have proved that the problem is locally well-posed for small initial data in $H^s(\mathbf{T})$ with $s \geq -\frac{j}{2} + 1, j \in \mathbb{N}^+$ with the aid of the standard Fourier restriction norm method. To the best of our knowledge, there is no result of well-posedness about the problem when $s < -\frac{j}{2} + 1$. In this paper, firstly, we prove that the bilinear estimate related to the nonlinear term of the equation in standard Bourgain space is invalid with $s < -\frac{j}{2} + 1$. Then we prove that the Cauchy problem for the periodic shallow water-type equation is locally well-posed in $H^s(\mathbf{T})$ with $-j + \frac{3}{2} < s < -\frac{j}{2} + 1, j \geq 2$ for arbitrary initial data. The novelty is that we introduce some new function spaces and give a useful relationship among new spaces.

1. Introduction

In this paper, we consider the Cauchy problem for the periodic shallow water type

equation

$$u_t + \partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) + \partial_x(1 - \partial_x^2)^{-1} \left[u^2 + \frac{1}{2}u_x^2 \right] = 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbf{T} = [0, 2\pi\lambda), \lambda \geq 1. \quad (1.2)$$

Obviously, (1.1) is the higher order modification of the Camassa-Holm equation

$$u_t + \frac{1}{2}\partial_x(u^2) + \partial_x(1 - \partial_x^2)^{-1} \left[u^2 + \frac{1}{2}u_x^2 \right] = 0 \quad (1.3)$$

in the nonlocal form. Equation (1.3) has been investigated by many people, for instance, see [4–11, 13–18, 25, 36, 45, 46].

Omitting the last term in (1.1) yields the higher order Korteweg-de Vries equation

$$u_t + \partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = 0. \quad (1.4)$$

Using the Fourier restriction norm method, Hirayama [21] proved that (1.1) is locally well-posed in $H^s(\mathbf{T})$ with $s \geq -\frac{j}{2}$. When $j = 1$, (1.3) reduces to the following KdV equation which possesses the bi-Hamiltonian structure and completely integrable and infinite conserved laws. Lots of people have investigated the Cauchy problem for the KdV equation, for instance, see [2, 3, 12, 21, 31–33, 39, 43]. Especially, Bourgain [2] introduced the Fourier restriction norm method which is an effective tool in solving the Cauchy problem for dispersive equations in low regularity, to establish the local well-posedness of the Cauchy problem for the KdV. Kenig et al. [31] proved that the Cauchy problem for the periodic KdV equation is locally well-posed in $H^s(\mathbf{T})$ with $s \geq -\frac{1}{2}$. Bourgain [3] proved that the Cauchy problem for the periodic KdV equation is ill-posed in $H^s(\mathbf{T})$ with $s < -\frac{1}{2}$ in the sense that the solution map is not C^3 . Colliander et al. [12] proved that the Cauchy problem for the periodic KdV equation is globally well-posed in $H^s(\mathbf{T})$ with $s \geq -\frac{1}{2}$ with the aid of I method. Recently, by using the inverse scattering method, Kappeler and Topalov [26] proved that the Cauchy problem for the KdV equation is globally well-posed in $H^s(\mathbf{T})$ with $s \geq -1$ in $H^s(\mathbf{T})$. Molinet [40] proved that the Cauchy problem for the KdV equation is ill-posed in $H^s(\mathbf{T})$ with $s < -1$. Many researchers have studied the non-periodic case of the KdV equation, for instance, see [20, 31–33].

Many people have investigated the periodic case and nonperiodic case of (1.1) [19, 22–24, 37, 38, 42, 44]. Himonas and Misiolek [22] have proved that the problem (1.1) is locally

well-posed for small initial data in $H^s(\mathbf{T})$ with $s \geq -\frac{j}{2} + 1, j \in N^+$ with the aid of the standard Fourier restriction norm method. Himonas and Misiolek [23] have proved that the problem (1.1) with $j = 1$ is locally well-posed for any initial data in $H^s(\mathbf{T})$ with $s \geq \frac{1}{2}$. To the best of our knowledge, there is no result about the well-posedness of (1.1) when initial data in $H^s(\mathbf{T})$ with $s < -\frac{j}{2} + 1, j \in N^+$. The main difficulty is that the structure of (1.1) is complicated. Recently, Yan et al. [48] proved that the problem (1.1) with $j = 1$ is locally well-posed for small initial data in $H^s(\mathbf{T})$ with $\frac{1}{6} < s < \frac{1}{2}$ with the aid of the new spaces. The spaces of (1.1) with $j \geq 2, j \in N$ are different from theirs of (1.1) with $j = 1$ due to different structure.

In recent ten years, to obtain low regularity of dispersive equations, some resolution function spaces have been introduced by some researchers [1, 27–30, 33–35]. Choosing a suitable function space is useful and difficult in dealing with the low regularity of dispersive equations, for instance, see [1, 20, 29, 30, 34]. In this paper, firstly, we prove that the bilinear estimate related to the nonlinear term of the equation in W^s defined below is invalid with $s < -\frac{j}{2} + 1$. Then, by introducing the new function spaces and the Strichartz estimate which are used to establish the bilinear estimates and using the fixed point Theorem, we prove that the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbf{T})$ with $-j + \frac{3}{2} < s < 1 - \frac{j}{2}$ with $j \geq 2, j \in Z$ for arbitrary initial data.

We give some notations before presenting the main results. $A \sim B$ means that $|B| \leq |A| \leq 4|B|$. $A \gg B$ means that $|A| \geq 4|B|$. C is a positive constant which may vary from line to line. $0 < \epsilon \ll 1$ means that $0 < \epsilon < \frac{1}{100j^5}$. Throughout this paper, $\dot{Z} := Z - \{0\}$. Denote dk by the normalized counting measure on \dot{Z} . $(dk)_\lambda$ the normalized counting measure on $\dot{Z}_\lambda = \frac{\dot{Z}}{\lambda}$:

$$\int a(k)(dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \dot{Z}_\lambda} a(k).$$

Denote $\mathcal{F}_x f$ by the Fourier transformation of a function f defined on $[0, 2\pi\lambda)$ with respect to the space variable

$$\mathcal{F}_x f(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi\lambda} e^{-ikx} f(x) dx.$$

and we have the Fourier inverse transformation formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \mathcal{F}_x f(k) (dk)_\lambda = \frac{1}{\sqrt{2\pi}} \sum_{k \in \dot{Z}} e^{ikx} \mathcal{F}_x f(k).$$

Denote $\mathcal{F}_t f$ by the Fourier transformation of a function f with respect to the time variable

$$\mathcal{F}_t f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-it\tau} f(t) dt$$

and we have the Fourier inverse transformation formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int e^{it\tau} \mathcal{F}_t f(\tau) d\tau.$$

Let

$$S(t)\phi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} e^{i(-1)^{j+1}k^{2j+1}} \mathcal{F}_x \phi(k) (dk)_\lambda.$$

We define the space-time Fourier transform $\mathcal{F}f(k, \tau)$ for $k \in \dot{Z}_\lambda$ and $\tau \in \mathbf{R}$ by

$$\mathcal{F}f(k, \tau) = \frac{1}{2\pi} \int \int_0^{2\pi\lambda} e^{-ikx} e^{-i\tau t} f(x, t) dx dt$$

and

$$f(x, t) = \frac{1}{2\pi} \int \int e^{ikx} e^{i\tau t} \mathcal{F}f(k, \tau) (dk)_\lambda d\tau.$$

Obviously, we have that

$$\begin{aligned} \|f\|_{L^2(\mathbf{T})} &= \|\mathcal{F}_x f\|_{L^2((dk)_\lambda)}, \\ \int_0^{2\pi} f(x) \bar{g}(x) dx &= \int \mathcal{F}_x f(k) \overline{\mathcal{F}_x g(k)} (dk)_\lambda, \\ \mathcal{F}_x(fg) &= \mathcal{F}_x f * \mathcal{F}_x g = \int \mathcal{F}_x f(k - k_1) \mathcal{F}_x g(k_1) (dk_1)_\lambda. \end{aligned}$$

Let

$$\begin{aligned} P(k) &= (-1)^{j+1} k^{2j+1}, \sigma = \tau - P(k), \quad \sigma_j = \tau_j - P(k_j), \\ D_1 &= \left\{ (\tau, k) \in \mathbf{R} \times \dot{Z} : |\tau - P(k)| \leq \frac{2j+1}{3} 4^{-j} |k|^{2j}, |k| \geq 1 \right\}, \\ D_2 &= \left\{ (\tau, k) \in \mathbf{R} \times \dot{Z} : \frac{2j+1}{3} 4^{-j} |k|^{2j} < |\tau - P(k)| < \frac{2j+1}{3} 4^{-j} |k|^{2j+1}, |k| \geq 1 \right\}, \\ D_3 &= \left\{ (\tau, k) \in \mathbf{R} \times \dot{Z} : |\tau - P(k)| \geq \frac{2j+1}{3} 4^{-j} |k|^{2j+1}, |k| \geq 1 \right\}, \\ D_4 &= \left\{ (\tau, k) \in \mathbf{R} \times \dot{Z} : |\tau - P(k)| > \frac{2j+1}{3} 4^{-j} |k|^{2j+1}, \frac{1}{\lambda} \leq |k| \leq 1 \right\}, \\ D_5 &= \left\{ (\tau, k) \in \mathbf{R} \times \dot{Z} : |\tau - P(k)| \leq \frac{2j+1}{3} 4^{-j} |k|^{2j+1}, \frac{1}{\lambda} \leq |k| \leq 1 \right\}, \\ \mathcal{F}(\Lambda^{-1})f &= \langle \sigma \rangle^{-1} \mathcal{F}f, \mathcal{F}J^s f = \langle k \rangle^s \mathcal{F}f(k). \end{aligned}$$

The Sobolev space $H^s(\mathbf{T})$ is defined by the following norm

$$\|f\|_{H^s(\mathbf{T})} = \|\langle k \rangle^s \mathcal{F}_x f(k)\|_{L^2(dk)_\lambda}$$

and define the $X_{s,b}$ spaces for 2π -periodic function via the norm

$$\|u\|_{X_{s,b}(\mathbf{T} \times \mathbf{R})} = \left\| \langle k \rangle^s \langle \sigma \rangle^b \mathcal{F} u(k, \tau) \right\|_{L^2((dk)_\lambda(d\tau))}.$$

The Z^s space is equipped with the following norm

$$\|u\|_{Z^s} = \|P_{D_1 \cup D_5} u\|_{X_{s, \frac{2j-1}{2j}}} + \|P_{D_2} u\|_{X_{(1-2j)(s-1), s}} + \|P_{D_3 \cup D_4} u\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} + \|u\|_{Y^s},$$

where $j \geq 2$ and $\|u\|_{Y^s} = \|\langle k \rangle^s \mathcal{F} u(k, \tau)\|_{L^2(dk)_\lambda L^1(d\tau)}$. Let

$$\|u\|_{W^s} = \|u\|_{X_{s, \frac{1}{2}}} + \|u\|_{Y^s}.$$

The main result of this paper are as follows.

Theorem 1.1. *Let $s < \frac{2-j}{2}$,*

$$F(u_1, u_2) = \frac{1}{2} \partial_x(u_1 u_2) + \partial_x(1 - \partial_x^2)^{-1} \left[u_1 u_2 + \frac{1}{2} (\partial_x u_1)(\partial_x u_2) \right],$$

and $u_j (j = 1, 2)$ be 2π -periodic functions. Then, we obtain that

$$\left\| \mathcal{F}^{-1} [\langle \tau + (-1)^j k^{2j+1} \rangle^{-1} \mathcal{F} F(u_1, u_2)] \right\|_{W^s} \leq C \prod_{j=1}^2 \|u_j\|_{W^s}$$

is untrue.

Remark 1. Theorem 1.1 implies that the standard Fourier restriction norm method is invalid when $s < 1 - \frac{j}{2}$. Lack of the bilinear estimates in W^s doesnot necessarily imply ill-posedness of problem. One can recover the bilinear estimates by changing new function spaces, for instance, see [1, 20, 29, 30]. Thus, by choosing suitable function spaces, we obtain the following Theorem 1.2.

Theorem 1.2. *Let $-j + \frac{3}{2} < s < -\frac{j}{2} + 1$, $j \geq 3$ and u_0 be $2\pi\lambda$ -periodic function. Then the Cauchy problems (1.1)(1.2) are locally well-posed in $H^s(\mathbf{T})$. More precisely, for any $u_0 \in H^s$ with $\|u_0\|_{H^s} \leq r$, there exists a solution $u \in C([-T, T]; H^s)$ to (1.1)(1.2) with $T = T(r) > 0$. Moreover, the solution is uniquely derived in Z_T^s embedded continuously into $C([-T, T]; H^s)$ and the data-to-solution map from $\{u_0 \in H^s \mid \|u_0\|_{H^s} \leq r\}$ to Z_T^s is Lipschitz.*

Remark 2. The optimal regularity indices of the Cauchy problem for (1.1) is unknown. We will pursue the optimal regularity indices of the Cauchy problem for (1.1). From Lemmas 2.1, 2.4 and the structure of (1.1), we choose the space $X_{s, \frac{2j-1}{2j}}$ related to D_1, D_5 . Since we consider the case $s < 1 - \frac{j}{2}$, we choose the space $X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}$ related to D_3, D_4 in view of high \times high \rightarrow low interaction. By a direct computation, we know that $X_{(1-2j)(s-1), s}$ related to D_2 is suitable.

The rest of the paper is arranged as follows. In Section 2, we present some preliminaries. In Section 3, we present some bilinear estimates. In Section 4, we present the proof of Theorem 1.1. In Section 5, we present the proof of Theorem 1.2.

2. Preliminaries

In this section, we give some preliminaries which are crucial in establishing Lemmas 3.1, 3.2 and Theorems 1.1, 1.2.

Lemma 2.1. *Let $u(x, t), v(x, t)$ be $2\pi\lambda$ -periodic functions and $a+b \geq \frac{j+1}{2j+1}$ and $\min\{a, b\} > \frac{1}{2(2j+1)}$. Then, we have that*

$$\|uv\|_{L_{xt}^2} \leq C\|u\|_{X_{0,a}(\mathbf{T} \times \mathbf{R})}\|v\|_{X_{0,b}(\mathbf{T} \times \mathbf{R})}, \quad (2.1)$$

$$\|uv\|_{X_{0,-a}} \leq C\|u\|_{X_{0,b}(\mathbf{T} \times \mathbf{R})}\|v\|_{L_{xt}^2}. \quad (2.2)$$

Lemma 2.1 can be proved similarly to Lemma 2.3 of [47].

Lemma 2.2. *Assume that $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon \in \mathbf{R}$ and $T > 0$. Then, we have that*

$$\|\eta(t)S(t)\phi\|_{Z^s} \leq C\|\phi\|_{H^s(\mathbf{T})}.$$

Proof. Combining the definition of Z^s with Lemma 2.3, we have that $X^{\frac{2j-1}{2j}} \hookrightarrow Z^s \hookrightarrow C([0, T] : H^s(\mathbf{T}))$.

We have completed the proof of Lemma 2.2.

Lemma 2.3. *Let $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$ and $T > 0$. Then, we have that*

$$\left\| \eta(t) \int_0^t S(t-\tau) \partial_x(uv) d\tau \right\|_{Z^s} \leq C \|\partial_x \Lambda^{-1}(uv)\|_{Z^s}.$$

For the proof of Lemma 2.3, we refer the readers to Lemma 2.3 of [34].

Lemma 2.4. *Let $-j + \frac{3}{2} + j\epsilon \leq 1 - \frac{j}{2} - j\epsilon$ and $j \geq 2, j \in \mathbb{Z}$. Then, we have that*

$$\|u\|_{X_{s, \frac{1}{2j}}} \leq C\|u\|_{Z^s} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}, \quad (2.3)$$

$$\|u\|_{X_{s, \frac{1}{2}}(D_1 \cup D_2)} \leq C\|u\|_{Z^s(D_1 \cup D_2)}. \quad (2.4)$$

Proof. We firstly prove that (2.3). When $\text{supp } \mathcal{F}u \subset D_1$, since $\frac{2j-1}{2j} \geq \frac{1}{2j}$, we have that $\|u\|_{X_{s, \frac{2j-1}{2j}}} \geq \|u\|_{X_{s, \frac{1}{2j}}}$. When $\text{supp } \mathcal{F}u \subset D_2$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that $\langle \sigma \rangle^{s-\frac{1}{2j}} \geq C\langle k \rangle^{2js-2j+1}$ which yields that $\langle k \rangle^s \langle \sigma \rangle^{\frac{1}{2j}} \leq C\langle k \rangle^{(1-2j)(s-1)} \langle \sigma \rangle^s$, thus, we have that $\|u\|_{X_{(1-2j)(s-1), s}} \geq \|u\|_{X_{s, \frac{1}{2j}}}$. When $\text{supp } \mathcal{F}u \subset D_3$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that $\langle \sigma \rangle^{\frac{s-1}{j} + \frac{2j-1}{2j}} \geq C\langle k \rangle^{s+1+\frac{s-1}{j}}$ which yields that $|k|^s \langle \sigma \rangle^{\frac{1}{2j}} \leq C\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1}$, thus, we have that $\|u\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \geq \|u\|_{X_{s, \frac{1}{2j}}}$. Consequently, we have that $\|u\|_{Z^s} \geq C\|u\|_{X_{s, \frac{1}{2j}}}$. When $\text{supp } \mathcal{F}u \subset D_2$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that $\langle \sigma \rangle^{-s+\frac{2j-1}{2j}} \geq C\langle k \rangle^{2js+1}$ which yields that $\langle k \rangle^{(1-2j)(s-1)} \langle \sigma \rangle^s \leq C\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}}$, thus, we have that $\|u\|_{X_{(1-2j)(s-1), s}} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}$. When $\text{supp } \mathcal{F}u \subset D_3$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that $\langle \sigma \rangle^{-\frac{s-1}{j}-\frac{1}{2j}} \geq C\langle k \rangle^{-s-1-\frac{s-1}{j}}$ which yields that $\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \leq C\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}}$, thus, we have that $\|u\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}$. Consequently, we have that $\|u\|_{Z^s} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}$. By Cauchy-Schwartz inequality with respect to τ , we have that $\|\langle k \rangle^s \mathcal{F}u\|_{l_k^2 l_\tau^1} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}$, consequently, we have that $\|u\|_{Z^s} \leq C\|u\|_{X_{s, \frac{2j-1}{2j}}}$. Now we prove (2.4). When $\text{supp } \mathcal{F}u \subset D_1$, since $\frac{2j-1}{2j} \geq \frac{1}{2}$, we have that $\|u\|_{X_{s, \frac{2j-1}{2j}}} \geq \|u\|_{X_{s, \frac{1}{2}}}$. When $\text{supp } \mathcal{F}u \subset D_2$, since $s \geq -j + \frac{3}{2} + j\epsilon$, we have that $\langle k \rangle^s \langle \sigma \rangle^{1/2} \leq C\langle k \rangle^{(1-2j)(s-1)} \langle \sigma \rangle^s$, consequently, we have that $\|u\|_{X_{(1-2j)(s-1), s}} \geq \|u\|_{X_{s, \frac{1}{2}}}$.

We have completed the proof of Lemma 2.4.

Lemma 2.5. *Let $k = k_1 + k_2$, $\tau = \tau_1 + \tau_2$ and*

$$\sigma = \tau - k^{2j+1}, \sigma_1 = \tau_1 - k_1^{2j+1}, \sigma_2 = \tau_2 - k_2^{2j+1}.$$

Then, we have that

$$3\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\sigma - \sigma_1 - \sigma_2| = |k^{2j+1} - k_1^{2j+1} - k_2^{2j+1}| \geq \frac{2j+1}{4j} |k_{\min}| |k_{\max}|^{2j}.$$

where

$$|k_{\min}| = \min\{|k|, |k_1|, |k_2|\}, |k_{\max}| = \max\{|k|, |k_1|, |k_2|\}.$$

Moreover, we have that one of three following cases must occur:

$$(a) : |\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{2j+1}{3} 4^{-j} |k_{\min}| |k_{\max}|^{2j}, \quad (2.5)$$

$$(b) : |\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{2j+1}{3} 4^{-j} |k_{\min}| |k_{\max}|^{2j}, \quad (2.6)$$

$$(c) : |\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{2j+1}{3} 4^{-j} |k_{\min}| |k_{\max}|^{2j}. \quad (2.7)$$

Proof of Lemma 2.5 can be seen in Lemma 2.4 of [38].

Lemma 2.6. *Let $Z = \mathbf{R}/2\pi\lambda$, $\lambda > 0$. Let $s \in \mathbf{R}$ and \mathcal{X}^s be a Banach space of functions on $\mathbf{R}_t \times Z$ with the following properties: (i) $\mathcal{S}(\mathbf{R} \times Z)$ is dense in \mathcal{X}^s , (ii) $X^{s,b}(\mathbf{R} \times Z) \hookrightarrow \mathcal{X}^s \hookrightarrow C_t(\mathbf{R}; H^s(Z))$ for some $b > \frac{1}{2}$, (iii) $X^{s',b'}(\mathbf{R} \times Z) \hookrightarrow \mathcal{X}^s$ for some $s' \in \mathbf{R}$ and $\frac{1}{2} \leq b' < 1$. Assume that $u \in \mathcal{X}^s$ satisfies $u(\cdot, 0) = 0$ in $H^s(Z)$. Then, we have*

$$\lim_{T \rightarrow +0} \|u\|_{\mathcal{X}_T^s} = 0. \quad (2.8)$$

For the proof of Lemma 2.6, we refer the readers to Proposition 2.6 of [34].

Remark 3. From Lemma 2.4, we have that $X_{s, \frac{2j-1}{2j}} \hookrightarrow Z^s \hookrightarrow Y^s$. It is easily checked that $Y^s \hookrightarrow C_t(\mathbf{R}; H^s(Z))$. Consequently, Z^s satisfies (i)-(iii).

3. Bilinear estimates

In this section, we present some crucial bilinear estimates. We always assume that $j \geq 2, j \in N^+$.

Lemma 3.1. *Let $j \geq 2$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$. Then, we have that*

$$\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}, \quad (3.1)$$

here $C > 0$, which is independent of λ , $\|\cdot\|_{X^s}$ is the norm removing $\|\cdot\|_{Y^s}$ from $\|\cdot\|_{Z^s}$.

Proof. Obviously, $(\mathbf{R} \times \dot{Z}_\lambda)^2 \subset \bigcup_{j=1}^8 \Omega_j$, where

$$\begin{aligned}\Omega_1 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 : \max\{|k_1|, |k|\} \leq 1 \right\}, \\ \Omega_2 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k_1| \sim |k_2| \gg |k| \geq 1 \right\}, \\ \Omega_3 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k_1| \sim |k_2| \gg |k|, 1 \geq |k| \geq \frac{1}{\lambda} \right\}, \\ \Omega_4 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k| \sim |k_2| \gg |k_1| \geq 1 \right\}, \\ \Omega_5 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k| \sim |k_2| \gg |k_1|, 1 \geq |k_1| \geq \frac{1}{\lambda} \right\}, \\ \Omega_6 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k| \sim |k_1| \gg |k_2| \geq 1 \right\}, \\ \Omega_7 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k| \sim |k_1| \gg |k_2|, 1 \geq |k_2| \geq \frac{1}{\lambda} \right\}, \\ \Omega_8 &= \left\{ (\tau_1, k_1, \tau, k) \in (\mathbf{R} \times \dot{Z}_\lambda)^2 \cap \Omega_1^c : |k| \sim |k_1| \sim |k_2| \geq 1 \right\}.\end{aligned}$$

(1) In region Ω_1 . By using Lemma 2.5 and the Young inequality, since $\max\{|k_1|, |k|\} \leq 1$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, from the definition of Z^s , we have that

$$\begin{aligned}\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} &\leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, \frac{2j-1}{2j}}} \\ &\leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2j}} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\ &\leq C \| |k| \|_{l_k^2} \| \mathcal{F} u_1 * \mathcal{F} u_2 \|_{l_k^\infty L_\tau^2} \leq C \| \mathcal{F} u_1 \|_{l_k^2 L_\tau^2} \| \mathcal{F} u_2 \|_{l_k^2 L_\tau^1} \leq C \| u_1 \|_{X_{s, \frac{1}{2j}}} \| u_2 \|_{Y^s} \\ &\leq C \prod_{j=1}^2 \| u_j \|_{Z^s}.\end{aligned}$$

(2) In region Ω_2 . In this region, we consider (a)-(c) of Lemma 2.5, respectively.

(a) Case $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$. In this case, we have that $\text{supp}[\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_3$.

When $\text{supp} \mathcal{F} u_j \subset \Omega_1 \cup \Omega_2$ with $j = 1, 2$, by using Lemmas 2.5, 2.3, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} &\leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} [(\langle k \rangle \mathcal{F} u_1) * (\langle k \rangle \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\ &\leq C \| (J^s u_1)(J^s u_2) \|_{L_{xt}^2} \leq C \| u_1 \|_{X_{s, \frac{1}{2}}} \| u_2 \|_{X_{s, \frac{1}{2(2j+1)} + \epsilon}} \\ &\leq C \| u_1 \|_{X_{s, \frac{1}{2}}} \| u_2 \|_{X_{s, \frac{1}{2j}}} \leq C \prod_{j=1}^2 \| u_j \|_{Z^s}.\end{aligned}$$

When $\text{supp } \mathcal{F}u_1 \subset \Omega_3$, by using Plancherel identity and the Hölder inequality as well as Lemma 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} [\langle k \rangle \mathcal{F}u_1 * (\langle k \rangle \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \mathcal{F}u_1 * [\langle k \rangle^{2s} \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F}u_1 \right] * [\langle k \rangle^{-2j} \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^{-2j+2} \mathcal{F}u_2\|_{l_k^1 L_\tau^1} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^{-2j+2-s} \langle k \rangle^s \mathcal{F}u_2\|_{l_k^1 L_\tau^1} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^s \mathcal{F}u_2\|_{l_k^2 L_\tau^1} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F}u_2 \subset \Omega_3$, this case can be proved similarly to $\text{supp } \mathcal{F}u_1 \subset \Omega_3$.

(b) Case $|\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, we consider the following cases:

$$(i) : |\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}, (ii) : |\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\},$$

respectively.

When (i) occurs: we consider $\text{supp } \mathcal{F}u_1 \subset D_1$, $\text{supp } \mathcal{F}u_1 \subset D_2$, respectively.

When $\text{supp } \mathcal{F}u_1 \subset D_1$ which yields that $|k| \leq C$, by using Lemmas 2.3, 2.5, 2.1, $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} (\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle \sigma \rangle^{-\frac{1}{2j}} \left[(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F}u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F}u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left(J^s \Lambda^{\frac{2j-1}{2j}} u_1 \right) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{-s-2j+3, \frac{1}{2}}} \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F}u_1 \subset D_2$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemmas 2.3, 2.5,

2.1, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [\langle k \rangle \mathcal{F} u_1 * \langle k \rangle \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{(1-2j)(s-1)} \Lambda^s u_1) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{(1-2j)(s-1), s}} \|u_2\|_{X_{-s-2j+3, \frac{1}{2}}} \\
& \leq C \|u_1\|_{X_{(1-2j)(s-1), s}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (ii) occurs: we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$ is valid, this case can be proved similarly to $|\sigma| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$ is valid, we consider $\text{supp } u_1 \subset D_1$, $\text{supp } u_1 \subset D_2$, $\text{supp } u_1 \subset D_3$, respectively.

When $\text{supp } u_1 \subset D_1$ which yields that $|k| \leq C$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemmas 2.3, 2.5, 2.1, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} (\langle k \rangle \mathcal{F} u_1 * \langle k \rangle \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \left\| (J^s \Lambda^{\frac{2j-1}{2j}} u_1) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{-s-2j+3, \frac{1}{2}}} \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } u_1 \subset D_2$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemmas 2.3, 2.5, 2.1, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [\langle k \rangle \mathcal{F} u_1 * (\langle k \rangle \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{(1-2j)(s-1)} \Lambda^s u_1) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{(1-2j)(s-1), s}} \|u_2\|_{X_{-s-2j+3, \frac{1}{2}}} \leq C \|u_1\|_{X_{(1-2j)(s-1), s}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F} u_1 \subset D_3$ which yields that $\mathcal{F} u_2 \subset D_3$, we consider $\text{supp } [\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_1$, $\text{supp } [\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_2$, $\text{supp } [\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_3$, respectively.

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, by using the Hölder inequality and the Young inequality and Lemma 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [|k| \mathcal{F}u_1 * |k| \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} \right) (\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2) \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \left(\langle k_1 \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F}u_1 \right) * \mathcal{F}u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s-3j+\frac{9}{2}+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.2}
\end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, by using the Hölder inequality and the Young inequality and Lemma 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{(1-2j)(s-1)+1} \langle \sigma \rangle^{s-1} [|k| \mathcal{F}u_1 * |k| \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{(1-2j)(s-1)+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{s-\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} \right) (\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2) \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \left(\langle k_1 \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F}u_1 \right) * \mathcal{F}u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s-3j+\frac{9}{2}+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.3}
\end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, by using the Hölder inequality and the Young inequality

and Lemma 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} [|k| \mathcal{F} u_1 * |k| \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{-\frac{s-1}{j} + \frac{1}{2} + \epsilon} \langle \sigma \rangle^{\frac{s-1}{j} + \frac{1}{2} + \epsilon} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} \right) (\langle k \rangle \mathcal{F} u_1) * (\langle k \rangle \mathcal{F} u_2) \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \left(\langle k_1 \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F} u_1 \right) * \mathcal{F} u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s-3j+\frac{9}{2}+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.4}
\end{aligned}$$

When (c) occurs: this case can be proved similarly to case (b).

(3) Region Ω_3 . We consider $|k| \leq |k_1|^{-2j}$ and $|k_1|^{-2j} < |k| \leq 1$, respectively.

When $|k| \leq |k_1|^{-2j}$, by using Lemma 2.3, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [|k| \mathcal{F} u_1 * |k| \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k_1|^{-3j+2} \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^2} \\
& \leq C \left\| [\langle k \rangle^{-(2j-1)} \mathcal{F} u_1 * (\langle k \rangle^{-(j-1)} \mathcal{F} u_2)] \right\|_{l_k^\infty L_\tau^2} \\
& \leq C \|u_1\|_{X_{1-2j,0}} \|u_2\|_{Y^{1-j}} \leq C \|u_1\|_{X_{1-2j,0}} \|u_2\|_{Y^s} \\
& \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.5}
\end{aligned}$$

Now we consider the case $|k_1|^{-2j} \leq |k| \leq 1$. In this case, we consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: in this case $\text{supp} [\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_4$, by using the Hölder inequality and the Young inequality, since $|k| \leq 1$ and $1 + \frac{s-1}{j} \geq 0$, by using Lemma 2.5, since

$-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{-\frac{s-1}{j}-3} \langle \sigma \rangle^{\frac{s-1}{j}} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k| [(|k|^s \mathcal{F} u_1) * (|k|^s \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (|k|^s \mathcal{F} u_1) * (|k|^s \mathcal{F} u_2) \right\|_{l_k^\infty L_\tau^2} \leq C \|u_1\|_{X_{s,0}} \|u_2\|_{Y^s} \\
& \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (b) occurs: we consider the case $|\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}$ and $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$, respectively.

When $|\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}$ is valid, we consider $\text{supp}[\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_4$ and $\text{supp}[\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_5$, respectively.

When $\text{supp}[\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_4$, by using the Hölder inequality and the Young inequality and Lemma 2.5, since $|k| \leq 1$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{-\frac{s-1}{j}-3} \langle \sigma \rangle^{\frac{s-1}{j}} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k|^{\frac{1}{2j}} \left[\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1 * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^2} \\
& \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{X_{s, \frac{2j-1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp}[\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_5$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{s-2} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle \sigma \rangle^{-\frac{1}{2j}} \left[(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left(J^s \Lambda^{\frac{2j-1}{2j}} u_1 \right) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|J^s \Lambda^{\frac{2j-1}{2j}} u_1\|_{L_{x_t}^2} \|J^{-s-2j+3} u_2\|_{X_{0, \frac{1}{2}}} \\
& \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{X_{s, \frac{2j-1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$ is valid, we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$, then this case can be proved similarly to case $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.
When $|\sigma_1| \sim |\sigma_2|$, we consider $\text{supp } \mathcal{F}u_1 \subset D_1$, $\text{supp } \mathcal{F}u_1 \subset D_2$, $\text{supp } \mathcal{F}u_1 \subset D_3$, respectively.

When $\text{supp } \mathcal{F}u_1 \subset D_1$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemma 2.1, 2.5, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{s-2} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left(J^s \Lambda^{\frac{2j-1}{2j}} u_1 \right) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\ & \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{-s-2j+3, \frac{1}{2}}} \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } \mathcal{F}u_1 \subset D_2$, by using Lemma 2.3, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$ and $|\sigma| \leq C|k_1|^{2j+1}$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{s-2} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| |k| \langle k \rangle^{s-\frac{3}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} \left[(|k|^{j+\frac{1}{2}-\frac{1}{2j}+(2j+1)\epsilon} \mathcal{F}u_1) * (|k| \mathcal{F}u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \left[(\langle k \rangle^{s+1-\frac{1}{2j}+j+(2j+2)\epsilon} \mathcal{F}u_1) * (\mathcal{F}u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \langle k \rangle^{-s-3j+3-\frac{1}{2j}+(2j+2)\epsilon} \prod_{j=1}^2 \left\| (J^{(1-2j)(s-1)} \Lambda^s u_j) \right\|_{l_k^2 L_\tau^2} \right\|_{l_k^\infty} \\ & \leq C \prod_{j=1}^2 \|u_j\|_{X_{(1-2j)(s-1), s}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}; \end{aligned}$$

When $\text{supp } \mathcal{F}u_1 \subset D_3$, we consider $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, respectively.

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, then, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| |k| \langle k \rangle^{s-2} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} [(\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s-4j+6} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{(1-2j)(s-1)-1} \langle \sigma \rangle^{s-1} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{(1-2j)(s-1)-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{s-\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} [(\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s-4j+6} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-2} \langle \sigma \rangle^{\frac{s-1}{j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-\frac{3}{2}+\epsilon} \langle \sigma \rangle^{\frac{s-1}{j}+\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{2s+j-3+(2j+2)\epsilon} [(\langle k \rangle \mathcal{F}u_1) * (\langle k \rangle \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s-4j+6} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (c) occurs: this case can be proved similarly to case (b).

(4) Region Ω_4 . We consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: we consider $|\sigma| > 4\max\{|\sigma_1|, |\sigma_2|\}$, $|\sigma| \leq 4\max\{|\sigma_1|, |\sigma_2|\}$, respectively.

When $|\sigma| > 4\max\{|\sigma_1|, |\sigma_2|\}$, then $\text{supp } \mathcal{F}u_2 \subset D_1 \cup D_2$ and $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$.

In this case, by using Lemmas 2.5, 2.3, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-(2j-1)(s-1)-1} \langle \sigma \rangle^{s-1} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \|(J^s u_1)(J^s u_2)\|_{L_{xt}^2} \\ & \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $|\sigma| \leq 4\max\{|\sigma_1|, |\sigma_2|\}$, we have that $|\sigma| \sim |\sigma_1|$ or $|\sigma| \sim |\sigma_2|$.

When $|\sigma| \sim |\sigma_1|$, we consider $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, respectively.

When $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, by using Lemmas 2.3, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq \|(J^s u_1)(J^s u_2)\|_{L_{xt}^2} \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, by using Lemmas 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \left(\prod_{j=1}^2 u_j \right) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-(2j-1)(s-1)-1} \langle \sigma \rangle^{s-1} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq \|(J^s u_1)(J^s u_2)\|_{L_{xt}^2} \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, by using Lemma 2.5, the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-2} \langle \sigma \rangle^{\frac{s-1}{j}} [|k| \mathcal{F}u_1 * |k| \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq \left\| (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_1) (J^{-2j} u_2) \right\|_{L_{xt}^2} \\ & \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|J^{-2j} u_2\|_{l_k^1 L_\tau^1} \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $|\sigma| \sim |\sigma_2|$, we consider $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, respectively.

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, $1 \leq |k_1| \leq C$, by using Lemmas 2.3, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1}(1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} &\leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ &\leq \|(J^s u_1)(J^s u_2)\|_{L_{xt}^2} \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, by using Lemma 2.5 and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1}(1 - \partial_x^2)^{-1} \partial_x \left(\prod_{j=1}^2 u_j \right) \right\|_{X^s} &\leq C \left\| \langle k \rangle^{-(2j-1)(s-1)-1} \langle \sigma \rangle^{s-1} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ &\leq \|(J^s u_1)(J^s u_2)\|_{L_{xt}^2} \leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp} [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, by using using Lemma 2.5 and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} &\leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-2} \langle \sigma \rangle^{\frac{s-1}{j}} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ &\leq \left\| (J^{-2j} u_1) (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_2) \right\|_{L_{xt}^2} \\ &\leq C \|J^{-2j} u_1\|_{l_k^1 L_\tau^1} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \|u_1\|_{Y^s} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

(b): $|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\}$. If $|\sigma_1| > 4 \max \{|\sigma|, |\sigma_2|\}$, then $\text{supp} \mathcal{F}u_1 \subset D_3$.

By using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} &\leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k_1| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ &\leq C \left\| (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_1) (J^{-s-(2j-3)} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\ &\leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $|\sigma_1| \leq 4 \max \{|\sigma|, |\sigma_2|\}$ we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$, this case can be proved similarly to case $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$, we consider $\text{supp } \mathcal{F}u_2 \subset D_2$ and $\text{supp } \mathcal{F}u_2 \subset D_3$, respectively.

When $\text{supp } \mathcal{F}u_2 \subset D_2$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq \left\| (J^{-s-(2j-2)} u_1) (J^{(1-2j)(s-1)} \Lambda^s u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\ & \leq C \|u_1\|_{Z^s(D_2 \cup D_3)} \|u_2\|_{X_{(1-2j)(s-1), s}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } \mathcal{F}u_2 \subset D_3$, we consider $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_3$, respectively.

When $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_1$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemmas 2.3, 2.5, 2.1, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k_1| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \langle k \rangle^{s+\frac{1}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [(|k_1| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \langle k \rangle^{s+j-\frac{1}{2}+(2j+1)\epsilon} [(|k_1| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \left[(\langle k \rangle^{-2s-2j+3} \mathcal{F}u_1) * (\langle k \rangle^{-s-j+\frac{3}{2}+(2j+1)\epsilon} \langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F}u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \left[(\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F}u_1) * (\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F}u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } [\mathcal{F}u_1 * \mathcal{F}u_2] \subset D_2$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq$

$1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{(1-2j)(s-1)-1} \langle \sigma \rangle^{s-1} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{(1-2j)(s-1)-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{s-\frac{1}{2}+\epsilon} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{s+j-\frac{3}{2}+(2j+1)\epsilon} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left[(\langle k \rangle^{-2s-2j+3} \mathcal{F} u_1) * (\langle k \rangle^{-s-j+\frac{1}{2}+(2j+1)\epsilon} \langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left[(\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_1) * (\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp} [\mathcal{F} u_1 * \mathcal{F} u_2] \subset D_3$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-2} \langle \sigma \rangle^{\frac{s-1}{j}} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}-\frac{3}{2}+\epsilon} \langle \sigma \rangle^{\frac{s-1}{j}+\frac{1}{2}+\epsilon} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{2s+j-3+(2j+2)\epsilon} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left[(\langle k \rangle^{-2s-2j+3} \mathcal{F} u_1) * (\langle k \rangle^{-j-1+(2j+2)\epsilon} \langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left[(\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_1) * (\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (c) occurs: we consider case $|\sigma_2| > 4\max\{|\sigma|, |\sigma_1|\}$, case $|\sigma_2| \leq 4\max\{|\sigma|, |\sigma_1|\}$, respectively.

When $|\sigma_2| > 4\max\{|\sigma|, |\sigma_1|\}$, obviously, $\text{supp} \mathcal{F} u_2 \subset D_2 \cup D_3$.

When $\text{supp} \mathcal{F} u_2 \subset D_2$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq \left\| (J^{-s-(2j-2)} u_1) (J^{(1-2j)(s-1)} \Lambda^s u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{Z^s} \|u_2\|_{X_{(1-2j)(s-1), s}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F}u_2 \subset D_3$ and $\text{supp } \mathcal{F}u_1 \subset D_1 \cup D_2$, by using Lemmas 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k_1| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| (J^{-s-(2j-3)} u_1) (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_2) \right\|_{X_{0,-\frac{1}{2j}}} \\ & \leq C \|u_1\|_{X_{s,\frac{1}{2}}} \|u_2\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } \mathcal{F}u_2 \subset D_3$ and $\text{supp } \mathcal{F}u_1 \subset D_3$, this case can be proved similarly to case $\text{supp } \mathcal{F}u_2 \subset D_3$ of $|\sigma_1| \sim |\sigma_2|$ in case (b).

Case $|\sigma_2| \leq 4\max\{|\sigma|, |\sigma_1|\}$ can be proved similarly to $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$.

(5) In region Ω_5 . In this region, we consider cases $|k_1| \leq |k|^{-2j}$ and $|k|^{-2j} < |k_1| \leq 1$, respectively.

When $|k_1| \leq |k|^{-2j}$, by using the Cauchy-Schwartz inequality and Young inequality as well as Lemma 2.3, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \langle k \rangle^{-2j} [\mathcal{F}u_1 * (\langle k \rangle^s \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| [\mathcal{F}u_1 * (\langle k \rangle^s \mathcal{F}u_2)] \right\|_{l_k^\infty L_\tau^2} \\ & \leq C \|\mathcal{F}u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \|\mathcal{F}u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned} \tag{3.6}$$

When $|k|^{-2j} \leq |k_1| \leq 1$, we consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: by using the Young inequality and Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F}u_1) * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left[(|k|^{-\frac{1}{2j}} \mathcal{F}u_1) * (\langle k \rangle^{s-1} \mathcal{F}u_2) \right] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| |k|^{-\frac{1}{2j}} \mathcal{F}u_1 \right\|_{l_k^1 L_\tau^2} \|u_2\|_{Y^s} \leq C \|\mathcal{F}u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned} \tag{3.7}$$

When (b) occurs: by using the Young inequality and Cauchy-Schwartz inequality, since

$-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[(|k|^{1-\frac{1}{2j}} \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{s-1} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k|^{1-\frac{1}{2j}} \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1 \right\|_{l_k^1 L_\tau^2} \|u_2\|_{Y^s} \\
& \leq C \|\langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \\
& \leq C \|u_1\|_{X_{0, \frac{1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.8}
\end{aligned}$$

When (c) occurs: by using the Young inequality and Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[(|k|^{1-\frac{1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^s \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k|^{1-\frac{1}{2j}} \mathcal{F} u_1 \right\|_{l_k^1 L_\tau^1} \|u_2\|_{X_{s, \frac{1}{2j}}} \\
& \leq C \|\mathcal{F} u_1\|_{l_k^2 L_\tau^1} \|u_2\|_{X_{s, \frac{1}{2j}}} \\
& \leq C \|u_1\|_{Y^s} \|u_2\|_{X_{s, \frac{1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.9}
\end{aligned}$$

(6) In region Ω_6 . This case can be proved similarly to Ω_4 .

(7) In region Ω_7 . This case can be proved similarly to Ω_5 .

(8) In region Ω_8 . We consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: we have that $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_3$.

If $|\sigma| > 4 \max\{|\sigma_1|, |\sigma_2|\}$ and $\text{supp} \mathcal{F} u_1 \subset D_1 \cup D_2$. In this case, by using Lemma 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \|(J^s u_1)(J^s u_2)\|_{L_{x,t}^2} \\
& \leq C \|u_1\|_{X_{s, \frac{1}{2}}} \|u_2\|_{X_{s, \frac{1}{2(2j+1)} + \epsilon}} \\
& \leq C \|u_1\|_{X_{s, \frac{1}{2}}} \|u_2\|_{X_{s, \frac{1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

If $|\sigma| > 4\max\{|\sigma_1|, |\sigma_2|\}$ and $\text{supp } \mathcal{F}u_1 \subset D_3$. In this case, by using Lemma 2.1 and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} (\mathcal{F}u_1 * \mathcal{F}u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq \left\| \left[J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_1 \right] [J^{-2j} u_2] \right\|_{L_{xt}^2} \\
& \leq \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^{-2j} \mathcal{F}u_2\|_{l_k^1 L_\tau^1} \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{Y^s} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

If $|\sigma| \leq 4\max\{|\sigma_1|, |\sigma_2|\}$, then we have that $|\sigma| \sim |\sigma_1|$ or $|\sigma| \sim |\sigma_2|$.

When $|\sigma| \sim |\sigma_1|$. In this case, we have that $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_3$. Since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemma 2.5 and the Young inequality, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} (\mathcal{F}u_1 * \mathcal{F}u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \mathcal{F}u_1 * [\langle k \rangle^{2s-2} \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[\langle k \rangle^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} \mathcal{F}u_1 \right] * [\langle k \rangle^{-2j} \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^{-2j} \mathcal{F}u_2\|_{l_k^1 L_\tau^1} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|\langle k \rangle^s \mathcal{F}u_2\|_{l_k^2 L_\tau^1} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $|\sigma| \sim |\sigma_2|$, this case can be proved similarly to case $|\sigma| \sim |\sigma_1|$.

When (b) occurs: if $|\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}$ which yields $\text{supp } \mathcal{F}u_1 \subset D_3$. In this case, we consider $\mathcal{F}u_2 \subset D_2 \cup D_3$. When $\mathcal{F}u_2 \subset D_2$, by using Lemma 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2j}} (\mathcal{F}u_1 * \mathcal{F}u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_1) (J^{-s-(2j-3)} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{s, \frac{1}{2}}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.10}
\end{aligned}$$

When $\mathcal{F}u_2 \subset D_3$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemmas 2.3, 2.5 and the

Young inequality, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F} u_1 \right) * \mathcal{F} u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s+\frac{9}{2}-3j+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s};
\end{aligned}$$

If $|\sigma_1| \leq 4 \max\{|\sigma|, |\sigma_2|\}$, we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$, this case can be proved similarly to $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$, we consider $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_1$, $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_2$, $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_3$, respectively.

When $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_1$, by using Lemmas 2.3, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{-\frac{1}{2j}+\frac{1}{2}+\epsilon} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F} u_1 \right) * \mathcal{F} u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s+\frac{9}{2}-3j+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp}(\mathcal{F} u_1 * \mathcal{F} u_2) \subset D_2$, by using Lemmas 2.3, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq$

$1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{(1-2j)(s-1)+1} \langle \sigma \rangle^{s-1} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{(1-2j)(s-1)+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{s-\frac{1}{2}+\epsilon} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{s+j+\frac{1}{2}+(2j+1)\epsilon} \mathcal{F}u_1 \right) * \mathcal{F}u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s+\frac{9}{2}-3j+(2j+1)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp}(\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_3$, by using Lemmas 2.3, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X^s} \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}} \langle \sigma \rangle^{\frac{s-1}{j}} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{-\frac{s-1}{j}+\frac{1}{2}+\epsilon} \langle \sigma \rangle^{\frac{s-1}{j}+\frac{1}{2}+\epsilon} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \left(\langle k \rangle^{2s+2j-2+(2j+2)\epsilon} \mathcal{F}u_1 \right) * \mathcal{F}u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-2s-2j+2+(2j+2)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (c) occurs: this case can be proved similarly to case (b).

We have completed the proof of Lemma 3.1.

Remark 4. Regions Ω_2 determines the indices $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$.

Lemma 3.2. *Let $j \geq 2$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$. Then, we have that*

$$\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \quad (3.11)$$

Proof. Obviously, $(\mathbf{R} \times \dot{Z}_\lambda)^2 \subset \bigcup_{j=1}^8 \Omega_j$, where $\Omega_j (1 \leq j \leq 8)$ are defined as Lemma 3.1.

(1) In region Ω_1 . By using the Lemma 2.3 and the Hölder inequality as well as the Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} &\leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, \frac{2j-1}{2j}}} \\ &\leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2j}} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \leq C \|k\|_{l_k^2} \|\mathcal{F} u_1 * \mathcal{F} u_2\|_{l_k^\infty L_\tau^2} \leq C \|\mathcal{F} u_1\|_{l_k^2 L_\tau^2} \|\mathcal{F} u_2\|_{l_k^2 L_\tau^1} \\ &\leq C \|u_1\|_{X_{s, \frac{1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

(2) In region Ω_2 . In this case, we consider case (a)-(c) of Lemma 2.5, respectively.

When (a) is valid, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, by using Lemma 2.5 and the Young inequality, we have that

$$\begin{aligned} \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} &\leq \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-1} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^1} \\ &\leq C \left\| (\langle k \rangle^{-j+1} \mathcal{F} u_1) * (\langle k \rangle^{-j+1} \mathcal{F} u_2) \right\|_{l_k^\infty L_\tau^1} \\ &\leq \left\| (\langle k \rangle^s \mathcal{F} u_1) * (\langle k \rangle^s \mathcal{F} u_2) \right\|_{l_k^\infty L_\tau^1} \leq C \prod_{j=1}^2 \|u_j\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When (b) is valid, we consider the following cases:

$$(i) : |\sigma_1| > 4 \max \{|\sigma|, |\sigma_2|\}, \quad (ii) : |\sigma_1| \leq 4 \max \{|\sigma|, |\sigma_2|\},$$

respectively.

When (i) occurs: we consider $\text{supp } u_1 \subset D_1$, $\text{supp } u_1 \subset D_2$, $\text{supp } u_1 \subset D_3$, respectively.

When $\text{supp } u_1 \subset D_1$ which yields that $|k| \leq C$, by using Lemmas 2.3, 2.1, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} &\leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\ &\leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2j}}} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} (|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\ &\leq C \left\| \langle \sigma \rangle^{-\frac{1}{2j}} (\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\ &\leq C \left\| \left(J^s \Lambda^{\frac{2j-1}{2j}} u_1 \right) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } u_1 \subset D_2$, by using Lemmas 2.3, 2.1, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\
& \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2j}}} \\
& \leq \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{(1-2j)(s-1)} \Lambda^s u_1) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{(1-2j)(s-1), s}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (ii) occurs: we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$ is valid, this case can be proved similarly to $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$, we consider $\text{supp } u_1 \subset D_1$, $\text{supp } u_1 \subset D_2$, $\text{supp } u_1 \subset D_3$, respectively.

When $\text{supp } u_1 \subset D_1$ which yields that $|k| \leq C$, by using Lemmas 2.3, 2.1, 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\
& \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2j}}} \leq \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2j}} [|k| \mathcal{F} u_1 * |k| \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle \sigma \rangle^{-\frac{1}{2j}} (\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{-s-2j+3} \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left(J^s \Lambda^{\frac{2j-1}{2j}} u_1 \right) (J^{-s-2j+3} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } u_1 \subset D_2$, we can assume that $\text{supp } u_2 \subset D_2$ and $|\sigma| \leq C|k_1|^{2j+1}$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$ and the Hölder inequality as well as the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq$

$s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\
& \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2} + \epsilon}} \\
& \leq \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2} + \epsilon} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2} + \epsilon} \langle \sigma \rangle^{2\epsilon} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| (\langle k \rangle^{2+(4j+2)\epsilon} \mathcal{F} u_1) * \mathcal{F} u_2 \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s+4-4j+(4j+2)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{(1-2j)(s-1), s}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{(1-2j)(s-1), s}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \tag{3.12}
\end{aligned}$$

(c) Case $|\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$. This case can be proved similarly to case (b).

(3) Region Ω_3 . We consider $|k| \leq |k_1|^{-2j}$ and $|k_1|^{-2j} \leq |k| \leq 1$, respectively.

When $|k| \leq |k_1|^{-2j}$, by using Lemma 2.3 and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \left[(\langle k \rangle^{-(\frac{3j}{2}-1)} \mathcal{F} u_1) * (\langle k \rangle^{-(\frac{3j}{2}-1)} \mathcal{F} u_2) \right] \right\|_{l_k^\infty L_\tau^2} \\
& \leq C \|u_1\|_{X_{1-\frac{3j}{2}, 0}} \|u_2\|_{Y^{1-2j}} \leq C \|u_1\|_{X_{1-\frac{3j}{2}, 0}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $|k_1|^{-2j} \leq |k| \leq 1$, we consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: by using the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| |k| \langle k \rangle^{s-2} \langle \sigma \rangle^{-1} [(|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| [(\langle k \rangle^{-j+1} \mathcal{F} u_1) * (\langle k \rangle^{-j+1} \mathcal{F} u_2)] \right\|_{l_k^\infty L_\tau^1} \\
& \leq C \prod_{j=1}^2 \|\langle k \rangle^{1-j} \mathcal{F} u_j\|_{l_k^2 L_\tau^1} \leq C \prod_{j=1}^2 \|u_j\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (b) occurs: we consider $|\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}$ and $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$, respectively.

When $|\sigma_1| > 4\max\{|\sigma|, |\sigma_2|\}$, $\text{supp } \mathcal{F}u_1 \subset D_1$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| |k|^{\frac{1}{2j}} \left(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F}u_1 \right) * \left(\langle k \rangle^{-s-2j+3} \mathcal{F}u_2 \right) \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F}u_1 \right) * \left(\langle k \rangle^{-s-2j+3} \mathcal{F}u_2 \right) \right\|_{l_k^\infty L_\tau^2} \\ & \leq C \|u_1\|_{X_{s, \frac{2j-1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$, we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$, this case can be proved similarly to case $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$, we consider $\text{supp } \mathcal{F}u_j \subset D_1$, $\text{supp } \mathcal{F}u_j \subset D_2$, $\text{supp } \mathcal{F}u_j \subset D_3$, respectively.

When $\text{supp } \mathcal{F}u_j \subset D_1$ with $j = 1, 2$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F}u_1 \right) * \left(\langle k \rangle^{-s-2j+3} \mathcal{F}u_2 \right) \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left(\langle k \rangle^s \langle \sigma \rangle^{\frac{2j-1}{2j}} \mathcal{F}u_1 \right) * \left(\langle k \rangle^{-s-2j+3} \mathcal{F}u_2 \right) \right\|_{l_k^\infty L_\tau^2} \\ & \leq C \prod_{j=1}^2 \|u_j\|_{X_{s, \frac{2j-1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } \mathcal{F}u_j \subset D_2$ with $j = 1, 2$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| |k| \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} [(|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left(\langle k \rangle^{(1-2j)(s-1)} \langle \sigma \rangle^s \mathcal{F}u_1 \right) * \left(\langle k \rangle^{-s-2j+3} \mathcal{F}u_2 \right) \right\|_{X^{0, -\frac{1}{2}+\epsilon}} \\ & \leq C \|u_1\|_{X^{(1-2j)(s-1), s}} \|u_2\|_{X^{-s-2j+3, \frac{1}{2}}} \\ & \leq C \|u_1\|_{X^{(1-2j)(s-1), s}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $\text{supp } \mathcal{F}u_j \subset D_3$ with $j = 1, 2$, we consider $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_1$, $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_2$, $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_3$, respectively.

When $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_1 \cup D_2$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2}+\epsilon}} \\
& \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} ((|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{2\epsilon} ((|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)) \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+(4j+1)\epsilon} ((|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)) \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s-4j+6} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } (\mathcal{F}u_1 * \mathcal{F}u_2) \subset D_3$, we consider $|\sigma| \leq C|k_1|^{2j+1}$ and $|\sigma| > C|k_1|^{2j+1}$, respectively.

When $|\sigma| \leq C|k_1|^{2j+1}$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2}+\epsilon}} \\
& \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} ((|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} \langle \sigma \rangle^{2\epsilon} ((|k| \mathcal{F}u_1) * (|k| \mathcal{F}u_2)) \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{s-\frac{1}{2}+\epsilon} (|k|^{1+4j\epsilon} \mathcal{F}u_1) * (|k| \mathcal{F}u_2) \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-4s-4j+4+4j\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $|\sigma| > C|k_1|^{2j+1}$, by using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, the Hölder inequality and the Young

inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^{s-1} \langle \sigma \rangle^{-1} (|k| \mathcal{F} u_1) * (|k| \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| \left((|k|^{-j+\frac{1}{2}} \mathcal{F} u_1) * (|k|^{-j+\frac{1}{2}} \mathcal{F} u_2) \right) \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| \left((|k|^{-j+\frac{1}{2}} \mathcal{F} u_1) \right) \right\|_{l_k^2 L_\tau^1} \left\| \left(|k|^{-j+\frac{1}{2}} \mathcal{F} u_2 \right) \right\|_{l_k^1 L_\tau^1} \\
& \leq C \left\| \left((|k|^{-j+\frac{1}{2}} \mathcal{F} u_1) \right) \right\|_{l_k^2 L_\tau^1} \left\| \langle k \rangle^s \left(|k|^{-j+\frac{1}{2}-s} \mathcal{F} u_2 \right) \right\|_{l_k^1 L_\tau^1} \\
& \leq C \prod_{j=1}^2 \left\| \left((|k|^{-j+\frac{1}{2}} \mathcal{F} u_j) \right) \right\|_{l_k^2 L_\tau^1} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

(c) Case $|\sigma_2| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\}$. This case can be proved similarly to case (b).

(4) Region Ω_4 . We consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: by using Lemma 2.5, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-1} [(|k| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| \langle k \rangle^{-2j} [\mathcal{F} u_1 * (\langle k \rangle^s \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| [(\langle k \rangle^{-2j} \mathcal{F} u_1) * (\langle k \rangle^s \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^1} \\
& \leq C \|\langle k \rangle^{-2j} u_1\|_{l_k^1 L_\tau^1} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

(b): $|\sigma_1| = \max \{|\sigma|, |\sigma_1|, |\sigma_2|\}$. In this case, we consider $|\sigma_1| > 4\max \{|\sigma|, |\sigma_2|\}$ and $|\sigma_1| \leq 4\max \{|\sigma|, |\sigma_2|\}$, respectively.

If $|\sigma_1| > 4\max \{|\sigma|, |\sigma_2|\}$, then $\text{supp } \mathcal{F} u_1 \subset D_3$ and $\text{supp } \mathcal{F} u_2 \subset D_1 \cup D_2$, by using Lemma 2.3, 2.5, 2.1, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \\
& \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2j}}} \\
& \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{-\frac{s-1}{j}-1} \langle \sigma \rangle^{\frac{s-1}{j}+1} u_1) (J^{-s-(2j-3)} u_2) \right\|_{X_{0, -\frac{1}{2j}}} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{s, \frac{1}{2}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $|\sigma_1| \leq 4\max\{|\sigma|, |\sigma_2|\}$, we have that $|\sigma_1| \sim |\sigma|$ or $|\sigma_1| \sim |\sigma_2|$.

When $|\sigma_1| \sim |\sigma|$, this case can be proved similarly to case $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$.

When $|\sigma_1| \sim |\sigma_2|$, we have that $\text{supp } \mathcal{F}u_1 \subset D_3$. In this case, we consider $\text{supp } \mathcal{F}u_2 \subset D_2, \text{supp } \mathcal{F}u_2 \subset D_3$, respectively.

When $\text{supp } \mathcal{F}u_2 \subset D_2$, by using Lemma 2.3 and $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \\
& \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2}+\epsilon}} \\
& \leq C \left\| \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} [(|k|^{-2j+2} \mathcal{F}u_1) * (\langle k \rangle^{(1-2j)(s-1)} \langle \sigma \rangle^s \mathcal{F}u_2)] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{-2j+2} u_1) * (J^{(1-2j)(s-1)} \Lambda^s u_2) \right\|_{X_{0, -\frac{1}{2}+\epsilon}} \\
& \leq C \|u_1\|_{X_{-2j+2, \frac{1}{2j}}} \|u_2\|_{X_{(1-2j)(s-1), s}} \\
& \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{(1-2j)(s-1), s}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F}u_j \subset D_3$ with $j = 1, 2$, without loss of generality, we can assume that $|\sigma| \leq C|k|^{2j+1}$ since $|\sigma| > C|k|^{2j+1}$ can be easily proved.

By using the Young inequality, by using Lemma 2.3 and $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\
& \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| \langle k \rangle^{s+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{2\epsilon} [\mathcal{F}u_1 * \mathcal{F}u_2] \right\|_{l_k^\infty L_\tau^\infty} \\
& \leq C \left\| (\mathcal{F}u_1) * (\langle k \rangle^{s+\frac{3}{2}+(4j+3)\epsilon} \mathcal{F}u_2) \right\|_{l_k^\infty l_\tau^\infty} \\
& \leq C \left\| \langle k \rangle^{-3s+\frac{11}{2}-4j+(4j+3)\epsilon} \right\|_{l_k^\infty} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned} \tag{3.13}$$

When case (c) occurs: by using Lemma 2.3, we have that

$$\left\| \Lambda^{-1} (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [|k| \mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2}.$$

By using a proof similar to case (c) of region Ω_4 of Lemma 3.1, we can obtain that

$$\left\| (1 - \partial_x^2)^{-1} \partial_x \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.$$

(5) In region Ω_5 . In this region, we consider $|k_1| \leq |k|^{-2j}$ and $|k|^{-2j} < |k_1| \leq 1$, respectively.

When $|k_1| \leq |k|^{-2j}$, by using Lemma 2.3 and the Young inequality as well as Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k_1| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \langle k \rangle^{-2j} [\mathcal{F} u_1 * (\langle k \rangle^s \mathcal{F} u_2)] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| [\mathcal{F} u_1 * (\langle k \rangle^s \mathcal{F} u_2)] \right\|_{l_k^\infty L_\tau^2} \leq C \|\mathcal{F} u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When $|k|^{-2j} \leq |k_1| \leq 1$, we consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: by using Lemma 2.3 and the Young inequality as well as Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^s \langle \sigma \rangle^{-\frac{1}{2j}} [(|k| \mathcal{F} u_1) * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left[(|k|^{1-\frac{1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^{s-1} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| |k|^{1-\frac{1}{2j}} \mathcal{F} u_1 \right\|_{l_k^1 L_\tau^2} \|u_2\|_{Y^{s-1}} \leq C \|\mathcal{F} u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

When (b) occurs: by using Lemma 2.3 and the Young inequality as well as the Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2j}} [\mathcal{F} u_1 * \mathcal{F} u_2] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| \left[(|k|^{-\frac{1}{2j}} \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^s \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\ & \leq C \left\| |k|^{-\frac{1}{2j}} \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1 \right\|_{l_k^1 L_\tau^2} \|u_2\|_{Y^s} \\ & \leq C \|\langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_1\|_{l_k^2 L_\tau^2} \|u_2\|_{Y^s} \leq C \|u_1\|_{X_{0, \frac{1}{2j}}} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned} \tag{3.14}$$

When (c) occurs: by using Lemma 2.3 and the Young inequality as well as the Cauchy-Schwartz inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x \left(\prod_{j=1}^2 u_j \right) \right\|_{X^s} \\
& \leq C \left\| \left[(|k|^{-\frac{1}{2j}} \mathcal{F} u_1) * (\langle k \rangle^s \langle \sigma \rangle^{\frac{1}{2j}} \mathcal{F} u_2) \right] \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| |k|^{-\frac{1}{2j}} \mathcal{F} u_1 \right\|_{l_k^1 L_\tau^1} \|u_2\|_{X_{s, \frac{1}{2j}}} \\
& \leq C \|\mathcal{F} u_1\|_{l_k^2 L_\tau^1} \|u_2\|_{Y^s} \leq C \|u_1\|_{Y^s} \|u_2\|_{X_{s, \frac{1}{2j}}} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

(6) In region Ω_6 . This case can be proved similarly to Ω_4 .

(7) In region Ω_7 . This case can be proved similarly to Ω_7 .

(8) In region Ω_8 . We consider (a)-(c) of Lemma 2.5, respectively.

When (a) occurs: by using Lemma 2.5 and the Young inequality, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$ as well as the Cauchy-Schwartz inequality, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-1} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^1} \\
& \leq C \left\| \langle k \rangle^{s-2j} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^1} \\
& \leq C \|\langle k \rangle^s \mathcal{F} u_1\|_{l_k^2 L_\tau^1} \|\langle k \rangle^{-2j} \mathcal{F} u_2\|_{l_k^1 L_\tau^1} \leq C \|u_1\|_{Y^s} \|u_2\|_{Y^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When (b) occurs: we have that $\text{supp } \mathcal{F} u_1 \subset D_3$.

When $\text{supp } \mathcal{F} u_2 \subset D_1 \cup D_2$, by using $X_{s, \frac{1}{2} + \epsilon} \hookrightarrow Y^s$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned}
& \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \\
& \leq C \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2} + \epsilon}} \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2} + \epsilon} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \\
& \leq C \left\| (J^{-\frac{s-1}{j}-1} \Lambda^{\frac{s-1}{j}+1} u_1) (J^{-s-(2j-3)} u_2) \right\|_{X_{0, -\frac{1}{2} + \epsilon}} \leq C \|u_1\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \|u_2\|_{X_{s, \frac{1}{2j}}} \\
& \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}.
\end{aligned}$$

When $\text{supp } \mathcal{F} u_j \subset D_3$ with $j = 1, 2$, without loss of generality, we can assume that $|\sigma| \leq C|k_1|^{2j+1}$ since case $|\sigma| > C|k_1|^{2j+1}$ can be easily proved.

By using $X_{s, \frac{1}{2}+\epsilon} \hookrightarrow Y^s$, since $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$, we have that

$$\begin{aligned} & \left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Y^s} \leq C \left\| \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{X_{s, -\frac{1}{2}+\epsilon}} \\ & \leq C \left\| \langle k \rangle^{s+1} \langle \sigma \rangle^{-\frac{1}{2}+\epsilon} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^2 L_\tau^2} \leq C \left\| \langle k \rangle^{s+\frac{3}{2}+\epsilon} \langle \sigma \rangle^{2\epsilon} (\mathcal{F} u_1 * \mathcal{F} u_2) \right\|_{l_k^\infty L_\tau^\infty} \\ & \leq C \left\| \langle k \rangle^{-3s-4j+\frac{11}{2}+(4j+2)\epsilon} \prod_{j=1}^2 \|u_j\|_{X_{-\frac{s-1}{j}-1, \frac{s-1}{j}+1}} \right\| \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \end{aligned}$$

Case (c) can be proved similarly to Case (b).

The proof of Lemma 3.2 is completed.

Remark 5. Regions Ω_3, Ω_4 are the most difficult to handle. Moreover, regions Ω_3, Ω_4 determine the indices $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$.

Lemma 3.3. *Let $j \geq 2$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$. Then, we have that*

$$\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (\partial_x u_j) \right\|_{Z^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \quad (3.15)$$

Proof. Combining the definition of Z^s with Lemmas 3.1, 3.2, we have Lemma 3.3.

We have completed the proof of Lemma 3.3.

By using a proof similar to Lemma 3.3, we have Lemmas 3.3, 3.4.

Lemma 3.4. *Let $j \geq 2$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$. Then, we have that*

$$\left\| \Lambda^{-1} \partial_x \prod_{j=1}^2 (u_j) \right\|_{Z^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \quad (3.16)$$

Lemma 3.5. *Let $j \geq 2$ and $-j + \frac{3}{2} + j\epsilon \leq s \leq 1 - \frac{j}{2} - j\epsilon$. Then, we have that*

$$\left\| \Lambda^{-1} \partial_x (1 - \partial_x^2)^{-1} \prod_{j=1}^2 (u_j) \right\|_{Z^s} \leq C \prod_{j=1}^2 \|u_j\|_{Z^s}. \quad (3.17)$$

4. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1.

Proof. We assume that $N \gg 1$, $a \in \dot{Z}$ and

$$\begin{aligned} \mathcal{F} u_1(k, \tau) &= (\chi_{(N)}(k) + \chi_{(N)}(-k)) \chi_{[-1,1]}(\tau + (-1)^k k^{2k+1}), \\ \mathcal{F} u_2(k, \tau) &= (\chi_{(1-N)}(k) + \chi_{(1-N)}(-k)) \chi_{[-1,1]}(\tau + (-1)^k k^{2k+1}), \end{aligned}$$

Where

$$\chi_a(k) = 1 \quad \text{if} \quad k = a, \chi_a(k) = 0 \quad \text{if} \quad k \neq a,$$

and

$$\chi_{[-1,1]}(\sigma) = 1 \quad \text{if} \quad |\sigma| \leq 1, \chi_{[-1,1]} = 0, \quad \text{if} \quad |\sigma| > 1.$$

Obviously, by a direct computation, we have that

$$\|u_j\|_{W^s} \sim N^s, j = 1, 2.$$

Let

$$\begin{aligned} R_1(k_1, k_2) &= \chi_N(k_1)\chi_{(1-N)}(k_2), R_2(k_1, k_2) = \chi_N(k_1)\chi_{(1-N)}(-k_2), \\ R_3(k_1, k_2) &= \chi_N(-k_1)\chi_{(1-N)}(k_2), R_4(k_1, k_2) = \chi_N(-k_1)\chi_{(1-N)}(-k_2). \end{aligned}$$

Then, we derive that

$$\begin{aligned} & \left\| \mathcal{F}^{-1} [\langle \tau + (-1)^j k^{2j+1} \rangle^{-1} \mathcal{F} F(u_1, u_2)] \right\|_{W^s} \\ = & \left\| \sum_{j=1}^4 \int_{\dot{Z}} \frac{|k|^{s+1}}{1+k^2} [k^2 + 3 + k_1 k_2] R_j(k_1, k_2) \left(\int_{\mathbf{R}} \langle \sigma \rangle^{-1/2} \chi_{[-1,1]}(\sigma_1) \chi_{[-1,1]}(\sigma_2) d\sigma_1 \right) dk_1 \right\|_{l_k^2 L_\sigma^2} \end{aligned}$$

By using Lemma 2.7, we obtain that

$$\langle \sigma \rangle \sim |k_{\min}| |k_{\max}|^{2j}$$

since $|\sigma_j| \leq 1$ with $j = 1, 2$. Thus, we have that

$$\int_{\mathbf{R}} \langle \sigma \rangle^{-1/2} \chi_{[-1,1]}(\sigma_1) \chi_{[-1,1]}(\sigma_2) d\sigma_1 \geq C |k_{\min}|^{-1/2} |k_{\max}|^{-j}.$$

By using a direct computation, we obtain that

$$\begin{aligned} & \left\| \mathcal{F}^{-1} [\langle \tau - k^3 \rangle^{-1} \mathcal{F} F(u_1, u_2)] \right\|_{W^s} \\ \geq & C \left\| \sum_{j=1}^4 \int_{\dot{Z}} \frac{|k|^{s+1}}{1+k^2} [k^2 + 3 + k_1 k_2] R_j(k_1, k_2) |k_{\min}|^{-1/2} |k_{\max}|^{-j} dk_1 \right\|_{l_k^2} \geq C N^{-j+2}. \end{aligned}$$

If (1.5) is untrue, then we have that

$$\begin{aligned} C N & \leq \left\| \mathcal{F}^{-1} [\langle \tau + (-1)^j k^{2j+1} \rangle^{-1} \mathcal{F} F(u_1, u_2)] \right\|_{X_{s, \frac{1}{2}}} \\ & \leq C \left\| \mathcal{F}^{-1} [\langle \tau + (-1)^j k^{2j+1} \rangle^{-1} \mathcal{F} F(u_1, u_2)] \right\|_{W^s} \leq C \prod_{j=1}^2 \|u_j\|_{W^s} \sim N^{2s}. \quad (4.1) \end{aligned}$$

Consequently, we obtain the contradiction since $s < -\frac{j}{2} + 1$. \square

We have completed the proof of Theorem 1.1.

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Now we are in a position to prove Theorem 1.2. Let $u^\mu := \mu^{-2j} u^\mu(\mu^{-1}x, \mu^{-2j-1}t)$.

Then, u^μ is the solution to the following problems

$$u_t^\mu + \partial_x^{2j+1} u^\mu + \frac{1}{2} \partial_x ((u^\mu)^2) + \partial_x (1 - \mu^2 \partial_x^2)^{-1} \left[(u^\mu)^2 + \frac{1}{2} \mu^2 (u_x^\mu)^2 \right] = 0, \quad (5.1)$$

$$u^\mu(x, 0) = \mu^{-2j} u_0(x/\mu) := u_0^\mu(x), \quad x \in \mathbf{T} = [0, 2\pi\lambda\mu) \quad (5.2)$$

if u is the solution to (1.1)-(1.2). Let

$$F^\mu(t) = \frac{1}{2} \partial_x (u^\mu)^2 + \partial_x (1 - \mu^2 \partial_x^2)^{-1} \left[(u^\mu)^2 + \frac{1}{2} \mu^2 (u_x^\mu)^2 \right].$$

We define

$$\Phi(u^\mu) = \eta(t) S(t) u_0^\mu(x) - \frac{1}{2} \eta(t) \int_0^t S(t-t') F^\mu(t') dt'. \quad (5.3)$$

We claim that for $\|u_0^\mu\|_{H^s} \leq r$, there exists $u^\mu \in Z_1^s$ satisfying

$$\Phi(u^\mu) = u^\mu. \quad (5.4)$$

By using Lemmas 2.2, 2.3, 3.3-3.5, we have that

$$\begin{aligned} \|\Phi(u^\mu)\|_{Z_1^s} &\leq \|\eta(t) S(t) u_0^\mu\|_{Z_1^s} + \left\| -\frac{1}{2} \eta(t) \int_0^t S(t-t') F^\mu(t') dt' \right\|_{Z_1^s} \\ &\leq C_1 \|u_0^\mu\|_{H^s(\mathbf{T})} + C \|\partial_x ((u^\mu)^2)\|_{Z_1^s} \leq C_1 \left[\|u_0^\mu\|_{H^s(\mathbf{T})} + (1 + \mu^2) \|u^\mu\|_{Z_1^s}^2 \right] \end{aligned}$$

and

$$\begin{aligned} \|\Phi(u^\mu) - \Phi(v^\mu)\|_{Z_1^s} &\leq C_1 (1 + \mu^2) \|u^\mu + v^\mu\|_{Z_1^s} \|u^\mu - v^\mu\|_{Z_1^s} \\ &\leq C_1 (1 + \mu^2) \left[\|u^\mu\|_{Z_1^s} + \|v^\mu\|_{Z_1^s} \right] \|u^\mu - v^\mu\|_{Z_1^s}. \end{aligned}$$

Let

$$B = \left\{ u \in Z_1^s : \|u\|_{Z_1^s} \leq (16C_1)^{-1} \mu^{-\frac{j}{2}-1}, j \geq 2 \right\}.$$

Thus, if $\mu^2 \geq \mu_0^2 := 16C_1$ and

$$\|u_0^\mu\|_{H^s(\mathbf{T})} \leq (16C_1)^{-2} \mu^{-j-\frac{1}{2}},$$

Φ will be a map from B to itself and Φ is a contraction map on B . Thus, the claim is valid. Consequently, there exists a solution to (5.3) for initial data ϕ^μ on the time interval $[-1, 1]$. By using a similar manner, we can obtain the Lipschitz continuity of the map Φ . Next, we consider (1.1)-(1.2) and $\|u_0\|_{H^s} \leq r$. If $r \leq (16C_1)^{-2}\mu_0^{\frac{1}{2}}$, then we derive that $\|u_0^{\mu_0}\|_{H^s} \leq \mu_0^{-j-1}\|u_0\|_{H^s} \leq (16C_1)^{-2}\mu_0^{-j-\frac{1}{2}}$ and derive a solution u^{μ_0} to the μ_0 -rescaled problem on $[0, 1]$, thus derive a solution u to (1.1) with existence time $T = \mu^{-2j-1}$. If $(16C_1)^{-2}\mu_0^{1/2} < r =: (16C_1)^{-2}\mu^{1/2}$, by using the same way, we solve the $\mu(r)$ -rescaled problem on $[-1, 1]$ to obtain a solution to (1.1) with $T = \mu(r)^{-2j-1}$.

Now we prove the uniqueness of the solution. Suppose that u_1 and u_2 are solutions to (1.1) with the common data u_0 and the common existence time T_0 and u and v belong to $Z_{T_0}^s$. Then, $u_k^\mu(x, t) = \mu^{-2j}u_k\left(\frac{x}{\mu^{2j+1}}, \frac{t}{\mu}\right)$ with $k = 1, 2$ are solutions to (1.1) corresponding to data $u_0^\mu = \mu^{-2j}u_0(\frac{x}{\mu^{2j+1}})$. By using Lemmas 2.3, 2.4, 3.3, we have that

$$\begin{aligned} & \|u_1^\mu - u_2^\mu\|_{Z_T^s} \\ & \leq C_1 \|u_1^\mu + u_2^\mu\|_{Z_T^s} \|u_1^\mu - u_2^\mu\|_{Z_T^s} \\ & \leq C_1 \left[\sum_{k=1}^2 \|u_k^\mu\|_{Z_T^s} \right] \|u_1^\mu - u_2^\mu\|_{Z_T^s} \end{aligned} \quad (5.5)$$

for $0 < T \leq \min\{1, \mu^{2j}T_0\}$. From Lemma 2.3, we have that

$$\begin{aligned} & \|u_k^\mu\|_{Z_T^s} \leq \|u_k^\mu - e^{(-1)^j \partial_x^{2j+1}} u_0^\mu\|_{Z_T^s} + \|u_0^\mu\|_{Z_T^s} \\ & \leq \|u_k^\mu - e^{(-1)^j \partial_x^{2j+1}} u_0^\mu\|_{Z_T^s} + \mu^{-2j}\|u_0\|_{H^s} \end{aligned} \quad (5.6)$$

with $k = 1, 2$. Combining $(u_k^\mu - e^{(-1)^j \partial_x^{2j+1}} u_0^\mu)_{t=0}$ with Lemma 2.6, we have that $\|u_k^\mu - e^{(-1)^j \partial_x^{2j+1}} u_0^\mu\|_{Z_T^s} \rightarrow 0$ as $T \rightarrow 0$. We can choose a sufficiently large $\mu = \mu(\|u_0\|_{H^s})$, such that

$$2C_1 C \mu^{-2j} \|u_0^\mu\|_{Z_T^s} \leq \frac{1}{4} \quad (5.7)$$

and a sufficiently small $T = T(\mu, u_1, u_2)$ such that

$$C_1 \left[\sum_{k=1}^2 \|u_k^\mu - e^{(-1)^j \partial_x^{2j+1}} u_0^\mu\|_{Z_T^s} \right] \leq \frac{1}{4}. \quad (5.8)$$

Combining (5.5)-(5.6) with (5.7)-(5.8), we have that

$$\|u_1^\mu - u_2^\mu\|_{Z_T^s} \leq \frac{1}{2} \|u_1^\mu - u_2^\mu\|_{Z_T^s}. \quad (5.9)$$

Consequently, we derive that $u_1 = u_2$ for $-\mu^{-2j}T \leq t \leq \mu^{-2j}T$. If $\mu^{-2j}T = T_0$, the conclusion is valid. If $\mu^{-2j}T < T_0$, by using a continuity argument which can be seen in [41], we can obtain the conclusion.

We have completed the proof of Theorem 1.2.

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